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Matrix Hamiltonians: SUSY approach to hidden symmetries

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Abstract. A new supersymmetric approach to dynamical symmetries for matrix quantum systems is explored. In contrast to standard one-dimensional quantum mechanics where there is no role for an additional symmetry due to nondegeneracy, matrix Hamiltonians allow nontrivial residual symmetries. This approach is based on a generalization of the intertwining relations familiar in SUSY quantum mechanics. The corresponding matrix supercharges, of first or of second order in derivatives, lead to an algebra which incorporates an additional block diagonal differential matrix operator (referred to as a 'hidden' symmetry operator) found to commute with the super-Hamiltonian. We discuss some physical interpretations of such dynamical systems in terms of spin $\frac{1}{2}$ particle in a magnetic field or in terms of coupled channel problem. Particular attention is paid to the case of transparent matrix potentials.

1. Introduction

Supersymmetric quantum mechanics (SUSY QM) [1] is an interesting framework for analysing nonrelativistic quantum problems. In particular it allows us to investigate the spectral properties of certain quantum models as well as to generate new systems with given spectral characteristics.

In general it is well known that SUSY algebra provides the relation between (super)partner Hamiltonians (which are often referred to as 'bosonic' and 'fermionic') associated to a two-fold degeneracy of levels.

Much less attention has been paid in literature to the possibility of using SUSY as a tool to study individual ('internal') symmetries of each superpartner Hamiltonian. In principle this can be useful if it allows us to integrate partially dynamical systems by discovering an additional dynamical symmetry which we shall refer to as a 'hidden' symmetry. Although we are unable to provide a straightforward procedure to reveal hidden symmetries of a given system in general, we will show that it is indeed possible to construct classes of examples of hidden symmetries by SUSY-inspired approaches*.

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* Notice that the existence of a symmetry operator is not necessarily associated with degeneracy of eigenstates of the Hamiltonian. One can convince oneself of this fact for matrix QM by taking 2×2 diagonal Hamiltonian with components having different spectra and no degeneracy. In this case a symmetry operator is obviously σ_3 playing the role of a grading operator.

The standard SUSY QM relations read

$$\{Q^+, Q^-\} = H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix} = \begin{pmatrix} q^+q^- & 0 \\ 0 & q^-q^+ \end{pmatrix} \tag{1}$$

$$h^{(i)} \equiv -\partial^2 + V^{(i)}(x) \qquad q^\pm = \mp\partial + W(x) \qquad \partial \equiv \partial/\partial x$$
$$Q^- = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix} \qquad Q^+ = \begin{pmatrix} 0 & q^+ \\ 0 & 0 \end{pmatrix} \tag{2}$$

$$\{Q^\pm, Q^\pm\} = 0 \tag{3}$$

$$[Q^\pm, H] = 0 \qquad h^{(1)}q^+ = q^+h^{(2)}. \tag{4}$$

In general there are different realizations of this algebra, for example multidimensional [2] and matrix [3] ones. It is also possible to generalize the algebra itself by preserving equations (3), (4) and allowing, by a nonstandard form of the intertwining operators $q^\pm$, to modify (1) to become

$$\{Q^+, Q^-\} = K = \begin{pmatrix} k^{(1)} & 0 \\ 0 & k^{(2)} \end{pmatrix} = \begin{pmatrix} q^+q^- & 0 \\ 0 & q^-q^+ \end{pmatrix} \tag{5}$$

where the diagonal operator K is no longer the super-Hamiltonian but has in general the nature of a symmetry operator

$$[k^{(i)}, h^{(i)}] = 0. \tag{6}$$

This generalization has already been discussed for one-dimensional [4] and two-dimensional [5] QM. For one-dimensional systems and intertwining operators of second order in derivatives the only relevant case was the one for which K is a function of H,

$$K = H^2 - 2\alpha H + \beta$$

where α and β are constants. For the two-dimensional case there exists the possibility of having a central charge R, which commutes with all elements of algebra, such that

$$K = f(H) + R. \tag{7}$$

A new, supersymmetrical, method was elaborated [4–5] to investigate hidden dynamical symmetries of quantum systems. The existence of such a differential operator R implies a dynamical symmetry (unknown *a priori*) made apparent by the generalized SUSY algebra.

Let us recall the physical impact of the supersymmetric approach to the Dirac equation [6] with some applications to superconductivity [7], to pseudorelativistic behaviour of electrons in two-band systems [8] and to attempts to a diagonalization procedure [3] in nuclear and atomic physics coupled channel problems and finally in the treatment of particles with spin in external magnetic fields [9]. Therefore it is important to discuss the role of symmetry operators for one-dimensional dynamical systems of Schrödinger type with matrix potentials. Formally the most straightforward method is to study the commutator of the Hamiltonian matrix with a generic differential operator matrix R and solve the corresponding system of differential equations obtained by imposing the commutator to vanish $[h, R] = 0$. Incidentally one can easily check that for the scalar case this system of equations (after possible subtraction of the Hamiltonian) does not allow a nontrivial solution. In contrast, in the matrix case nontrivial solutions exist even for the case of symmetry operators of first order in derivatives. For higher-order derivatives the equations become rather cumbersome and it is too difficult to provide the general discussion of the solutions.

For this higher-order case a method of solution suggested by supersymmetry seems to be useful. It starts from the same idea of factorization originally proposed by Schrödinger [10] with the related intertwining operators (see (4)) but now it is not applied to the

Hamiltonian but rather to the symmetry operator: this method reduces the order of differential equations one must solve without, however, increasing their number. In this paper we shall find nontrivial genuine second-order operators R even for one-dimensional first-order intertwining matrix operators.

The paper is organized as follows. In section 2 we shall investigate the SUSY approach with intertwining operators of first order in derivatives. A variety of matrix systems allowing genuine second-order symmetry operators will be obtained. In section 3 this method will be applied to a more complex case of higher-order intertwining operators, examining in particular the conditions for the factorizability of these intertwining operators (reducibility). The type of physical systems which we can describe in our formalism include a particle with magnetic moment in a magnetic field and more generally a class of systems with one continuous and one discrete degrees of freedom. A novel construction is given for transparent matrix potentials which are not duplications of standard scalar transparent potentials and are not generated by iterations of first-order Darboux transformations.

2. First-order matrix SUSY QM

We start from the general first-order (in derivatives) representation of the components of supercharges in the case of one-dimensional QM

$$q^+ \equiv A\partial + \tilde{B} \qquad q^- \equiv -A^\dagger\partial + \tilde{B}^\dagger \tag{8}$$

where A and $\tilde{B}$ are matrices. Imposing that these operators intertwine the Hamiltonians $h^{(1)}$ and $h^{(2)}$ (see equation (4)) reads

$$(-\partial^2 + V^{(1)}(x))(A\partial + \tilde{B}) = (A\partial + \tilde{B})(-\partial^2 + V^{(2)}(x)) \tag{9}$$

where $V^{(1)}, V^{(2)}$ are Hermitian potential matrices. Equation (9) amounts to solve the following three equations:

$$A' = 0 \tag{10}$$

$$V^{(1)}A - AV^{(2)} = 2\tilde{B}' \tag{11}$$

$$-\tilde{B}'' + V^{(1)}\tilde{B} - \tilde{B}V^{(2)} - AV^{(2)\prime} = 0. \tag{12}$$

The first equation implies that hereafter we will assume A to be a constant matrix. As usual in the frameworks of SUSY QM, the intertwining relations equation (9) lead to the connection between the column eigenfunctions of Hamiltonians $h^{(1)}$ and $h^{(2)}$:

$$\Psi^{(1)}(x) = (A\partial + \tilde{B})\Psi^{(2)}(x) \qquad \Psi^{(2)}(x) = (A\partial + \tilde{B})^\dagger\Psi^{(1)}(x). \tag{13}$$

Equation (13) sometimes allows zero modes, and the spectra of the partner Hamiltonian coincides up to these zero modes. What is not standard is the fact that equation (1) does not hold because the products of the supercharge components are no longer equal to the Hamiltonians:

$$q^+q^- = k^{(1)} = -AA^\dagger\partial^2 + (A\tilde{B}^\dagger - \tilde{B}A^\dagger)\partial + \tilde{B}\tilde{B}^\dagger + A\tilde{B}^{\dagger\prime} \tag{14}$$

$$q^-q^+ = k^{(2)} = -A^\dagger A\partial^2 + (\tilde{B}^\dagger A - A^\dagger\tilde{B})\partial + \tilde{B}^\dagger\tilde{B} - A^\dagger\tilde{B}'. \tag{15}$$

It is natural to consider $\det A \neq 0$ and $\det A = 0$ separately because the way to solve the system of equations (10)–(12) differs for the two cases. In the case $\det A \neq 0$ it is possible to subtract the Hamiltonians $h^{(1)}$ and $h^{(2)}$ from $k^{(1)}$ and $k^{(2)}$, being left with a symmetry operators of first order in derivatives; otherwise we deal with second-order operators. Furthermore, by a suitable similarity transformation induced by the matrix A

itself it is possible to consider $A = -1$ as a representative of the case $\det A \neq 0$ whereas we can take the matrix $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ as a representative of $\det A = 0$.

2.1. The case $A = -1$, symmetry operators of first order

In this case equations (11) and (12) become

$$V^{(2)} - V^{(1)} = 2\tilde{B}' \tag{16}$$

$$-\tilde{B}'' + V^{(1)}\tilde{B} - \tilde{B}V^{(2)} + V^{(2)\prime} = 0. \tag{17}$$

One can parametrize the potential matrices taking into account (16):

$$V^{(1)} = \tilde{B}^2 - \tilde{B}' + W \qquad V^{(2)} = \tilde{B}^2 + \tilde{B}' + W. \tag{18}$$

Then equation (17) becomes simply

$$W'(x) = [\tilde{B}(x), W(x)]. \tag{19}$$

The reader familiar with ordered products, for example in gauge field theory, will not have difficulty recognizing that equation (19) has not a simple local solution unless

$$[\tilde{B}(x), \tilde{B}(y)] = 0.$$

Since effectively this last condition reduces the matrix problem to a scalar problem we do not find this case of any interest to us.

From (16) one deduces that the non-Hermitian part of $\tilde{B}$ does not depend on x because of the hermiticity of the potentials. Therefore we can parametrize $\tilde{B}(x) = B(x) + \mathrm{i}C$, where B and C are both Hermitian matrices and C is a constant matrix. Correspondingly one has to solve the system of matrix equations

$$W' = [B, W] + \mathrm{i}[C, W] \tag{20}$$

$$W - W^\dagger = -2\mathrm{i}\{B, C\} \tag{21}$$

which are the consequence of (17) and of the hermiticity of both potentials. Restricting oneself to 2×2 matrix potentials, one can study the system of equations (20) and (21) by expanding all matrices in Pauli matrices and unity:

$$B = b_0 + b_i\sigma_i \qquad C = c_0 + c_i\sigma_i \qquad W = w_0 + w_i\sigma_i \tag{22}$$

with components b_0, b_i and c_0, c_i real. The related symmetry operators read:

$$R^{(1)} = q^+q^- - h^{(1)} = 2\mathrm{i}C\partial + 2C^2 - 2\mathrm{i}BC - W \tag{23}$$

$$R^{(2)} = q^-q^+ - h^{(2)} = 2\mathrm{i}C\partial + 2C^2 - 2\mathrm{i}CB + W. \tag{24}$$

While it is not restrictive to set $c_1 = c_2 = 0$, it is not interesting to choose c_3 and c_0 both to vanish. Indeed in this case the symmetry operators above are no longer differential operators and become proportional to a constant matrix W.

A general solution of the nonlinear system of matrix equations (20), (21) amounts first to finding a solution of the subsystem:

$$\begin{aligned}
&\mathrm{Im}\, w_0 = -2(b_0c_0 + b_3c_3) \\
&\mathrm{Im}\, w_1 = -2b_1c_0 \\
&\mathrm{Im}\, w_2 = -2b_2c_0 \\
&\mathrm{Im}\, w_3 = -2(b_3c_0 + b_0c_3) \\
&w_0 = \text{constant} \\
&\mathrm{Re}\, w_3 = \text{constant}
\end{aligned}$$

however, a complete solution of (20), (21) cannot be written. Particular solutions of (20) and (21) in terms of components (22) can be obtained by making specific ansatze.

(1) Let us assume

$$c_0 = b_3 = \mathrm{Re}\, w_3 = 0 \qquad \mathrm{Im}\, w_3(x) \equiv 0 \qquad c_3 \neq 0$$

$b_3 \neq 0$ would correspond to a trivial solution $W(x) =$ constant. Then one obtains

$$b_0 = 0 \qquad \mathrm{Re}\, w_1 = \beta \cos(2c_3 x)$$
$$\mathrm{Re}\, w_2 = -\beta \sin(2c_3 x) \qquad b_2(x) = -b_1(x) \tan(2c_3 x)$$

where β is a constant parameter and $b_1(x) \equiv \tilde{b}_1(x) \cos(2c_3 x)$ with $\tilde{b}_1(x)$ an arbitrary nonsingular function. Correspondingly the potentials read

$$\begin{aligned} V^{(1),(2)}(x) = \tilde{b}_1^2(x) - c_3^2 + \mathrm{Re}\, w_0 + [\beta \cos(2c_3x) \mp \tilde{b}_1'(x) \cos(2c_3x) \pm 2c_3\tilde{b}_1 \sin(2c_3x)] \cdot \sigma_1 \\ -[\beta \sin(2c_3x) \mp \tilde{b}_1'(x) \sin(2c_3x) \mp 2c_3\tilde{b}_1(x) \cos(2c_3x)] \cdot \sigma_2 \end{aligned} \tag{25}$$

and the symmetry operators (23), (24) after subtraction of constants become:

$$\begin{aligned} R^{(1),(2)} = \mathrm{i}\sigma_3\partial \mp \left[\tilde{b}_1(x) \sin(2c_3x) + \frac{\beta}{2c_3} \cos(2c_3x)\right] \cdot \sigma_1 \\ \mp \left[\tilde{b}_1(x) \cos(2c_3x) - \frac{\beta}{2c_3} \sin(2c_3x)\right] \cdot \sigma_2. \end{aligned} \tag{26}$$

Among the different interpretations concerning the physics of the matrix potentials $V^{(1),(2)}$ one consists of a spin $\frac{1}{2}$ neutral particle in a (inhomogeneous) magnetic field. It is necessary to assume that the magnetic field depends only on the coordinate $x \equiv x_3$ and lies in the (x_1, x_2)-plane in order to ensure the vanishing of its divergence $\partial_i B_i = 0$. While the motion in the (x_1, x_2)-plane is trivially free, the dynamics is still rather interesting because of the x_3 motion [9]. Physics-wise the inhomogeneity of the magnetic field is determined in this case by the requirement of vanishing of the scalar potential in (25) (neutrality of the particle).

(2) Let us assume

$$c_0 = b_3 = \mathrm{Re}\, w_3 = 0 \qquad \mathrm{Im}\, w_3(x) \neq 0 \qquad c_3 \neq 0$$

as before $b_3 \neq 0$ would correspond to a trivial solution $W(x) =$ constant. Then one obtains

$$b_0(x) = -\frac{1}{2c_3} \mathrm{Im}\, w_3(x) \qquad b_1(x) = \frac{(\mathrm{Re}\, w_2)' + 2c_3 \mathrm{Re}\, w_1}{2 \,\mathrm{Im}\, w_3}$$
$$b_2(x) = \frac{-(\mathrm{Re}\, w_1)' + 2c_3 \mathrm{Re}\, w_2}{2 \,\mathrm{Im}\, w_3}$$

where

$$\mathrm{Re}\, w_1(x) = \sqrt{(\mathrm{Im}\, w_3)^2 + \beta} \cos f(x) = \sqrt{(2b_0c_3)^2 + \beta} \cos f(x)$$
$$\mathrm{Re}\, w_2(x) = \sqrt{(\mathrm{Im}\, w_3)^2 + \beta} \sin f(x) = \sqrt{(2b_0c_3)^2 + \beta} \sin f(x).$$

β is a constant parameter as well as c_3 and $f(x)$ and $\mathrm{Im}\, w_3(x)$ are arbitrary functions. The potentials and the symmetry operators read:

$$\begin{aligned} V^{(1),(2)}(x) = b_0^2 \mp b_0' + b_1^2 + b_2^2 - c_3^2 + \mathrm{Re}\, w_0 + [2b_0b_1 + \mathrm{Re}\, w_1 \mp b_1'] \cdot \sigma_1 \\ +[2b_0b_2 + \mathrm{Re}\, w_2 \mp b_2'] \cdot \sigma_2 \end{aligned} \tag{27}$$

$$R^{(1),(2)} = \mathrm{i}\sigma_3\partial \pm \left[b_2(x) - \frac{\mathrm{Re}\, w_1}{2c_3}\right] \cdot \sigma_1 \pm \left[-b_1(x) - \frac{\mathrm{Re}\, w_2}{2c_3}\right] \cdot \sigma_2. \tag{28}$$

In terms of the 'magnetic' interpretation given above one can now notice that the absence of the scalar potential in (27) is less restrictive because the magnetic field still depends on one arbitrary function. The intrinsic 'periodicity' of the magnetic field forces a similar periodicity of the wavefunction. We warn, however, not to interpret the periodicity too naively since it depends in general on the properties of the arbitrary function $f(x)$, for example asymptotically constant magnetic field can be incorporated in this scheme.

One can also find solutions for other ansatze such as, for example

$$c_0 = 0 \qquad c_3 \neq 0 \qquad \mathrm{Re}\, w_3 = \text{constant} \neq 0$$

similarly to (1) and (2).

2.2. The case $det A = 0$, the symmetry operators of second order

The constant (see (10)) matrix A now has the form

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}. \tag{29}$$

We write the matrices $V^{(1),(2)}$ and B explicitly and for simplicity we assume them to be real:

$$V^{(i)} = \begin{pmatrix} v_1^{(i)} & v^{(i)} \\ v^{(i)} & v_2^{(i)} \end{pmatrix} \tag{30}$$

$$B(x) = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}. \tag{31}$$

Equations (11) and (12) can be solved and the potentials can be written in a similar way as before in terms of Pauli matrices, however the reality condition forces the absence of a σ_2 term †. The hidden symmetry operators are now of second order in derivatives.

In the case $b \neq 0$ it is possible to find several solutions dependent on arbitrary functions; in the case $b = 0, a = 1$ the form of the solutions simplifies. One solution is given by:

$$\begin{aligned} v_1^{(1)}(x) &= b_1' + b_2^2 + c^2 b_2^{-2} + c_1 \qquad v^{(1)}(x) = -2cb_2'b_2^{-2} \\ v_2^{(1)}(x) &= -\left(\frac{b_2'}{b_2}\right)' + \left(\frac{b_2'}{b_2}\right)^2 + \frac{2b_2'b_1}{b_2} + b_1^2 + b_2^2 + \frac{c^2}{b_2^2} - b_1' + \tilde{c}_1 \\ v_1^{(2)}(x) &= -b_1' + b_2^2 + \frac{c^2}{b_2^2} + c_1 \qquad v^{(2)}(x) = -2b_2' \\ v_2^{(2)}(x) &= \frac{b_2''}{b_2} + \frac{2b_2'b_1}{b_2} + b_1^2 + b_2^2 + \frac{c^2}{b_2^2} - b_1' + \tilde{c}_1 \end{aligned} \tag{32}$$

where $b_1(x), b_2(x)$ are arbitrary functions, $b_3(x) = cb_2^{-1}(x)$ and $b_4 = 0$. The symmetry operators $R^{(1),(2)}$ can be straightforwardly derived according to (14), (15) and are second-order differential operators.

For the second solution $v_1^{(1),(2)}(x)$ and $v^{(1),(2)}(x)$ are the same as in (32) and

$$v_2^{(1)}(x) = \frac{b_3''}{b_3} - \frac{2b_3'b_1}{b_3} - \frac{2b_2'b_4}{b_3} - b_1' + \sum_{k=1}^{4} b_k^2 + \tilde{c}_1$$

$$v_2^{(2)}(x) = \frac{b_2''}{b_2} + \frac{2b_2'b_1}{b_2} + \frac{2b_3'b_4}{b_2} + b_1' + \sum_{k=1}^{4} b_k^2 + \tilde{c}_1$$

† The interpretation of the dynamics as a magnetic interaction of $s = \frac{1}{2}$ particle is now still possible but we can also consider it as a coupled channel problem as discussed in [3].

where $b_2(x), b_3(x)$ are arbitrary functions such that $b_2 b_3 \neq 0,\ b_4 = \text{constant} \neq 0$ and

$$b_1(x) = (b_2 b_3)^{-1} \left[\frac{(b_2 b_3)^2}{2b_4} - (2b_4)^{-1}(b_2^2 + b_3^2) + \alpha \right]$$

with α a constant parameter.

3. Second-order matrix SUSY QM

Let us define the second-order differential operators

$$q^+ = (q^-)^\dagger = \partial^2 - 2F(x)\partial + B(x) \tag{33}$$
$$q^- = (q^+)^\dagger = \partial^2 + 2F^\dagger(x)\partial + B^\dagger(x) + 2F'^\dagger(x) \tag{34}$$

where $F(x)$ and $B(x)$ are 2×2 matrices. This representation for $q^\pm$ can be inserted into (2)–(5).

The intertwining relations are equivalent to a system of three nonlinear matrix differential equations:

$$V^{(1)} - V^{(2)} + 4F' = 0 \tag{35}$$
$$F'' - V^{(1)}F + FV^{(2)} - B' - V^{(2)\prime} = 0 \tag{36}$$
$$B'' + V^{(2)\prime\prime} - V^{(1)}B + BV^{(2)} - 2FV^{(2)\prime} = 0. \tag{37}$$

Our attitude towards the solution of this system of equations is that we consider $q^\pm, h^{(2)}, h^{(1)}$ to be essentially unknown except for the Schrödinger form of Hamiltonians and assumption of structure (33) of the supercharges, so the problem is to find the solution in terms of the matrices $F(x), B(x), V^{(i)}(x)$.

Due to the complexity of the problem (matrix, second derivatives, nonlinearity) it does not seem realistic to search for a general solution in analytic form, instead we believe that techniques of higher-order SUSY QM as developed in [4] can provide a useful tool for solving in an 'indirect' way by the ansatz of factorizability of $q^\pm$, i.e. restricting to the *reducible* matrix higher-order SUSY QM.

Another possibility which we mention is a particular solution for which the terms $FV^{(2)} - V^{(1)}F$ appearing in equation (36) reduce to $\{F, F'\}$ with the aim of a 'direct' integration of this equation. A sufficient condition which allows this integration is

$$V^{(1)} + V^{(2)} = 2P(F)$$

where P is an arbitrary 'scalar' function of the matrix $F(x)$ such as for example $P = \Sigma c_n(x) F^n(x)$ where c_n are scalar functions.

3.1. Reducible higher-order matrix SUSY QM

A specific ansatz consists of the factorizability of the operators $q^\pm$ of (33), (34) in terms of ordinary superpotentials $W(x)$ and $\tilde{W}(x)$:

$$q^+ = q_1^+ q_2^+ = (-\partial + W(x))(-\partial + \tilde{W}(x)) \tag{38}$$

connected by the ladder equation

$$q_1^- q_1^+ = q_2^+ q_2^- + \hat{\Delta} \qquad \text{or} \qquad W' + W^2 = -\tilde{W}' + \tilde{W}^2 + \hat{\Delta} \tag{39}$$

with $\hat{\Delta}$ being a constant Hermitian matrix as will be clear later on. Let us assume furthermore $\hat{\Delta}$ to be diagonal†. Then F and B of equations (33) and (34) are determined by the superpotentials $W(x)$, $\tilde{W}(x)$:

$$2F = W + \tilde{W} \qquad B = W\tilde{W} - \tilde{W}'. \tag{40}$$

The factorization equation (38) arises from two successive standard SUSY QM transformations

$$\begin{pmatrix} h^{(1)} & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} q_1^+ q_1^- & 0 \\ 0 & q_1^- q_1^+ \end{pmatrix} \tag{41}$$

and

$$\begin{pmatrix} h & 0 \\ 0 & h^{(2)} + \hat{\Delta} \end{pmatrix} = \begin{pmatrix} q_2^+ q_2^- + \hat{\Delta} & 0 \\ 0 & q_2^- q_2^+ + \hat{\Delta} \end{pmatrix} \tag{42}$$

by deleting the 'intermediate' Hamiltonian:

$$H = \begin{pmatrix} q_1^+ q_1^- & 0 \\ 0 & q_2^- q_2^+ + \hat{\Delta} \end{pmatrix}. \tag{43}$$

The matrix $\hat{\Delta}$ has to be such that $[H, Q^{\pm}] = 0$ and consequently $[q_2^{\pm}, \hat{\Delta}] = 0$ which makes the reason clear why $\hat{\Delta}$ has to be constant and that $[\tilde{W}, \hat{\Delta}] = 0$. We can therefore conclude that $[h^{(2)} + \hat{\Delta}, \hat{\Delta}] = 0$ and this allows us to identify $R^{(2)} \equiv \hat{\Delta}$ as a *symmetry operator* for $h^{(2)}$. If it further commutes with $h^{(1)}$ this operator is such that

$$\{Q^+, Q^-\} = (H)^2 - \hat{\Gamma} \cdot H \tag{44}$$

where Γ is a block diagonal matrix

$$\Gamma \equiv \begin{pmatrix} \hat{\Delta} & 0 \\ 0 & \hat{\Delta} \end{pmatrix} \tag{45}$$

and thus it corresponds to a rather 'trivial' R operator. Hereafter we shall exclude such a case imposing that the operator

$$R^{(1)} = -q_1^+ \hat{\Delta} q_1^- \tag{46}$$

is nontrivial ‡. The last case is rather interesting because it incorporates the possibility of having genuine partner symmetry operators of different orders in the derivatives.

In order to derive a solution we expand the previous operators in terms of Pauli matrices:

$$W(x) = w_0 + w_i\sigma_i \qquad \tilde{W}(x) = \tilde{w}_0 + \tilde{w}_3\sigma_3 \qquad \hat{\Delta} = \delta_0 + \delta_3\sigma_3 \qquad \delta_3 \neq 0. \tag{47}$$

To illustrate the techniques involved we now give an example.

Example 1.

$$\delta_0 = 0 \qquad w_0(x) = \tilde{w}_0(x) \qquad w_2(x) = 0.$$

Equation (39) then takes the form

$$\begin{aligned} &2w_0w_3 + w_3' - 2w_0\tilde{w}_3 + \tilde{w}_3' - \delta_3 = 0 \\ &2w_0w_1 + w_1' = 0 \\ &w_3^2 + w_1^2 + 2w_0' - \tilde{w}_3^2 = 0 \end{aligned} \tag{48}$$

with the following solution

$$w_0 = \tilde{w}_0 = 1/(2x) \qquad w_1 = 1/x \qquad w_3 = \tilde{w}_3 = \delta_3 x/2. \tag{49}$$

† If it is not, it can be diagonalized by a constant unitary transformation which also affects other operators.
‡ Equation (46) first appeared in the concluding section of [2].

In order to arrive at an interpretation of these results we have to write the potentials:

$$V^{(1)}(x) = W^2 - W' = 7/4x^2 + \delta_3^2 x^2/4 + (2/x^2)\cdot\sigma_1 \tag{50}$$

$$V^{(2)}(x) = W^2 + W' + \hat{\Delta} = -1/(4x^2) + \delta_3^2 x^2/4 + 2\delta_3\cdot\sigma_3. \tag{51}$$

These potentials contain centrifugal-like singularities and therefore we restrict the eigenvalue problem (radial problem) on the semiaxis $x > 0$. The physical solutions regular at the origin (a_i, b_i are constants) L_2-normalizable behave as:

$$\Psi^{(1)}(x) \underset{x\to 0}{\sim} \begin{pmatrix} a_1x^{5/2} + a_2x^{1/2} \\ a_1x^{5/2} - a_2x^{1/2} \end{pmatrix} \qquad \Psi^{(2)}(x) \underset{x\to 0}{\sim} \begin{pmatrix} b_1x^{1/2} \\ b_2x^{1/2} \end{pmatrix}. \tag{52}$$

Both potentials for $\delta_3 \neq 0$ lead to a discrete spectrum.

The symmetry operator $R^{(1)}$ can be calculated from equation (46) and is found to also contain σ_2-type terms:

$$R^{(1)} = \sigma_3\partial^2 + (2\mathrm{i}/x)\sigma_2\partial + \{-\delta_3\cdot\sigma_1 - (\mathrm{i}/x^2)\cdot\sigma_2 + [1/(4x^2) - \delta_3^2x^2/4]\cdot\sigma_3\}. \tag{53}$$

This operator also contains centrifugal singularities but one can easily prove that it maps L_2-normalizable solutions (52) into regular and normalizable solutions. Therefore it is a true symmetry operator. The partner symmetry operator $R^{(2)} = \delta_3\sigma_3$ is regular.

As in section 2.1 this example also allows an interpretation in terms of the external field as a magnetic field. It is possible to consider a magnetic field as a (pseudo)vector in a plane orthogonal to the one-dimensional axis in which the particle is allowed to move $x \equiv x_2$†. In contrast to section 2.1, the magnetic field in $V^{(2)}$ is homogeneous along the x_3-axis, while in $V^{(1)}$ it is *not* homogeneous (depends on x_2) and has nonzero components in the (x_1, x_3) plane.

To describe other examples it is useful to introduce

$$W(x) + \tilde{W}(x) \equiv 2F(x) = 2f_0 + 2f_i\sigma_i \qquad 2f_1 = w_1 \qquad 2f_2 = w_2$$

then (39) becomes:

$$2F' - 4F^2 + 2\{F, W\} = \hat{\Delta}. \tag{54}$$

The problem is now expressed in terms of a *matrix nonlinear differential* equation (54) for F, W. Since we are unable to present a general (analytic) discussion we provide particular solutions of the system (54) which in terms of components can be rewritten:

$$2f_0' - 4(f_0)^2 + 4(f_1)^2 + 4(f_2)^2 - 4(f_3)^2 + 4f_0w_0 + 4f_3w_3 = \delta_0 \tag{55}$$

$$f_1' + 2w_0f_1 = 0 \tag{56}$$

$$f_2' + 2w_0f_2 = 0 \tag{57}$$

$$2f_3' - 8f_0f_3 + 4f_0w_3 + 4w_0f_3 = \delta_3. \tag{58}$$

The solutions presented in the following will allow us to construct explicitly the partner potentials and the symmetry operators by using the general expressions in terms of only the f_k, $(k = 0, 1, 2, 3)$ and w_0, w_3:

$$V^{(1)}(x) = w_0^2 + w_3^2 - w_0' + 4(f_1^2 + f_2^2) - 4f_1'\cdot\sigma_1 - 4f_2'\cdot\sigma_2 + (2w_0w_3 - w_3')\cdot\sigma_3$$

$$V^{(2)}(x) = w_0^2 + w_3^2 - w_0' + 4(f_1^2 + f_2^2 + f_0') - \delta_0 + [2w_0w_3 - w_3' + 8(2f_0f_3 - f_0w_3 - f_3w_0) + \delta_3]\cdot\sigma_3.$$

† This choice will allow us to implement the condition of absence of sources of magnetic field automatically, $\partial_i B_i = 0$.

It is advantageous to not use expression (46) directly but to define a symmetry operator by suitable subtraction and rescaling (see section 1):

$$\tilde{R}^{(1)} \equiv -\frac{R^{(1)} + \delta_0 h^{(1)}}{\delta_3} = -\sigma_3\partial^2 + 4\mathrm{i}(f_2\sigma_1 - f_1\sigma_2)\partial$$
$$+(2w_0w_3 - w_3') + 4(w_3 f_1 - \mathrm{i}w_0 f_2)\cdot\sigma_1 + 4(w_3 f_2 + \mathrm{i}w_0 f_1)\cdot\sigma_2$$
$$+(w_0^2 + w_3^2 - 4f_1^2 - 4f_2^2 - w_0')\cdot\sigma_3$$

and we remind that $R^{(2)} \equiv \hat{\Delta}$.

We now list two particular cases

Example 2.

$$f_0 \equiv 0 \qquad f_3 \not\equiv 0.$$

The solution is of the type

$$2f_1(x) = \gamma_1 \exp\left(-2\int w_0 \mathrm{d}x\right)$$
$$2f_2(x) = \gamma_2 \exp\left(-2\int w_0 \mathrm{d}x\right)$$
$$2f_3(x) = \exp\left(-2\int w_0 \mathrm{d}x\right)\left[\gamma_3 + \delta_3 \exp\left(+2\int w_0 \mathrm{d}x\right)\right]$$
$$w_3(x) = (4f_3)^{-1}\left[\delta_0 - (\gamma_1^2 + \gamma_2^2)\exp\left(+4\int w_0 \mathrm{d}x\right) + 4f_3^2\right].$$

Example 3.

$$f_3 \equiv 0 \qquad f_0 \not\equiv 0 \qquad \delta_0 = 0.$$

The solution of the equations (55)–(58) can be written as

$$2f_1(x) = \gamma_1 \exp\left(-2\int w_0 \mathrm{d}x\right)$$
$$2f_2(x) = \gamma_2 \exp\left(-2\int w_0 \mathrm{d}x\right)$$
$$w_3(x) = \delta_3/(4f_0)$$
$$f_0(x) = \tfrac{1}{2}\sqrt{\gamma_1^2 + \gamma_2^2}\exp\left(-2\int w_0 \mathrm{d}x\right)\tanh\left(-\sqrt{\gamma_1^2 + \gamma_2^2}\int \exp\left(-2\int w_0 \mathrm{d}x\right) + C\right).$$

3.2. *Transparent matrix potentials*

We now explore a physical case for which the Hamiltonian $H^{(2)}$ describes free motion; in equations (35)–(37) we assume $V^{(2)} = 0$ and, as a consequence of SUSY, $V^{(1)}$ becomes a so-called transparent matrix potential [4, 3]:

$$V^{(1)} + 4F' = 0 \tag{59}$$
$$F'' - V^{(1)}F - B' = 0 \tag{60}$$
$$B'' - V^{(1)}B = 0. \tag{61}$$

These relations allow further simplifications by eliminating $V^{(1)}$ and B in terms of F using equation (59) and differentiating equation (60) to obtain B'' which is then introduced in equation (61) leading to a linear algebraic equation for B†. Inserting the expression for the derivative of B back into equation (60) we obtain

$$F'' + 4F'F + [(4F')^{-1}(F''' + 4F'^2 + 4F''F)]' = 0 \tag{62}$$

or equivalently

$$(F'' + 4F'F)' + 4F' \int (F'' + 4F'F)\,\mathrm{d}x = 0 \tag{63}$$

a nonlinear fourth-order equation for F whose solution may allow us to identify a class of transparent potentials in a 2×2 coupled channel problem.

A sufficient condition for F to satisfy equation (62) is given by the *simpler equation of second order*

$$(F'' + 4F'F) = \gamma F' \tag{64}$$

with γ being an arbitrary constant number. It easy to verify the property that if $F(x)$ is a solution of the nonlinear matrix equation (64) then $\tilde{F}(x) \equiv F(x) + \alpha \cdot I$ with $\alpha =$ constant is again a solution of the same equation but for the shifted value of $\tilde{\gamma} = \gamma + 4\alpha$. This peculiar property therefore allows us to restrict to the case $\gamma = 0$ in (64). The solution of this equation, because of (59), can be searched for, by parametrizing $F(x) \equiv G(x) + \mathrm{i}C$ with $G(x)$ and C Hermitian and C a constant matrix. We have thus to solve

$$G''(x) + 4G'(x)G(x) + 4\mathrm{i}G'(x)C = 0. \tag{65}$$

We can expand G and C in Pauli matrices, by a suitable rotation we can choose $c_1 = c_2 = 0$ and furthermore we assume $c_0 = 0$. Then we arrive at a system of equations:

$$\begin{aligned}
&g_0' + 2g_0^2 + 2\vec{g}^{\,2} = 2\gamma_0 \\
&g_i' + 4g_0 g_i - 4\epsilon_{ij3} c_3 g_j = 2\gamma_i \qquad i, j = 1, 2 \\
&\epsilon_{ijk} g_j' g_k + g_0' c_3 \delta_{3i} = 0 \qquad i, j, k = 1, 2, 3 \\
&g_3 = 2\gamma_3
\end{aligned}$$

with constant γ's and obvious meaning of g's and c's.

A solution can be found for all $\gamma_\mu = 0$:

$$\begin{aligned}
g_0(x) &= \frac{1}{4x + \beta} \\
g_1(x) &= g_0(x) \cos\phi(x) \\
g_2(x) &= g_0(x) \sin\phi(x) \\
\phi(x) &\equiv -4c_3 x + \zeta
\end{aligned}$$

with β and ζ arbitrary real constants.

By shifting x by $\beta/4$ and ζ by $c_3\beta$ we obtain the following expression for the transparent Hermitian matrix potential $V^{(1)}(x)$ with a singularity at the origin and long-range behaviour:

$$\begin{aligned}
V^{(1)}(x) &= \frac{1}{x^2} + \left[\frac{1}{x^2}\cos\phi(x) - \frac{4c_3}{x}\sin\phi(x)\right] \cdot \sigma_1 + \left[\frac{1}{x^2}\sin\phi(x) + \frac{4c_3}{x}\cos\phi(x)\right] \cdot \sigma_2 \\
&= \frac{1}{x^2}(1 + \hat{N}(x)) - \frac{4\mathrm{i}c_3}{x}\sigma_3 \hat{N}(x)
\end{aligned} \tag{66}$$

† Incidentally we note that equation (61) is a Schrödinger equation for B with eigenvalue zero. This is related to the fact that $V^{(2)} = 0$ and therefore there exists a trivial zero-energy solution, namely the constant wavefunction. Letting A^+ act on this wavefunction one obtains the corresponding one for $h^{(1)}$ with the same zero energy as proportional to the matrix $B(x)$ acting on a constant vector.

where

$$\hat{N}(x) \equiv \sigma_n \cos(4c_3 x) + \mathrm{i}\sigma_3\sigma_n \sin(4c_3 x)$$
$$\sigma_n \equiv \sigma_1 \cos\zeta + \sigma_2 \sin\zeta.$$

Due to the centrifugal singularity in this potential

$$V^{(1)}(x) \underset{x\to 0}{\sim} \frac{1}{x^2}(1+\sigma_n) + \mathcal{O}(1)$$

one should consider the scattering problem on the semiaxis $x \geqslant 0$.

To analyse the behaviour of wavefunctions at the origin it is useful to choose as a basis the following matrices: $\sigma_n, \tilde{\sigma}_n \equiv (\sigma_2 \cos\zeta - \sigma_1 \sin\zeta\ )$ and σ_3 with nonstandard realization:

$$\sigma_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tilde{\sigma}_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Then potential $V^{(1)}(x)$ becomes diagonal at the origin, but of course not in general, and its wavefunctions for $x \to 0$ are:

$$\Psi^{(1)}(x) \underset{x\to 0}{\sim} \begin{pmatrix} ax^2 \\ bx \end{pmatrix}.$$

The dynamical symmetry operator for $h^{(1)}$ can be derived from

$$\begin{aligned}\tilde{\tilde{R}}^{(1)} &\equiv q^+q^- - (h^{(1)})^2 = -4\mathrm{i}C\partial^3 + (-4G^2 - 4C^2 - 4\mathrm{i}[C,G] - 2G')\cdot\partial^2 \\ &\quad -(8GG' + 8\mathrm{i}CG' + 2G'')\partial - (2G''' + 4GG'' + 4\mathrm{i}CG'' + 16G'^2).\end{aligned}$$

This operator is also singular at the origin:

$$\begin{aligned}\tilde{\tilde{R}}^{(1)} &\underset{x\to 0}{\sim} -4\mathrm{i}c_3\sigma_3\partial^3 + \left(\frac{4c_3}{x}\tilde{\sigma}_n - 4c_3^2 + 16c_3^2\sigma_n - 32c_3^3 x\tilde{\sigma}_n\right)\cdot\partial^2 \\ &\quad + \left(\frac{4\mathrm{i}c_3}{x^2}\sigma_3 - \frac{4c_3}{x^2}\tilde{\sigma}_n - 32c_3^3\tilde{\sigma}_n\right)\cdot\partial \\ &\quad + \left(-\frac{4\mathrm{i}c_3}{x^3}\sigma_3 + \frac{4c_3}{x^3}\tilde{\sigma}_n - \frac{12c_3^2}{x^2} - \frac{12c_3^2}{x^2}\sigma_n + \frac{32c_3^3}{x}\tilde{\sigma}_n + 92c_3^4\sigma_n\right).\end{aligned}$$

Nevertheless it maps wavefunctions regular at the origin into regular ones similarly to example 1 (section 3.1).

Proceeding in the same way for $\gamma_0 > 0, \gamma_k = 0$ $(k = 1, 2, 3)$ we find another solution:

$$g_0(x) = \omega\coth(4\omega x)$$
$$g_1(x) = \frac{\omega\cos\phi}{\sinh(4\omega x)}$$
$$g_2(x) = \frac{\omega\sin\phi}{\sinh(4\omega x)}$$

with $\phi(x)$ as before and $\omega \equiv \sqrt{\gamma_0}$. In this case the corresponding potential $V^{(1)}(x)$ has the same centrifugal behaviour at the origin. The analogous analysis of wavefunctions and of their transformations by symmetry operator $R^{(1)}$ can be performed.

One can check that $F(x)$ does not factorize the space dependence from a constant matrix. We also stress that in general $V^{(1)}(x)$ cannot be made diagonal by global rotation and therefore it is not to be viewed as a pair of standard scalar reflectionless potentials: this means that there is flux from one channel to the other.

In order to ascertain the reducible or irreducible character of the solutions of (64) it is important to clarify the conditions for reducibility for the case $V^{(2)} = 0$ and $V^{(1)}(x)$ a transparent potential. As a consequence of (39), (41), (42) these conditions read:

$$V^{(1)} = W^2 - W' = -4F' \tag{67}$$

$$V^{(2)} = \tilde{W}^2 + \tilde{W}' + \hat{\Delta} = 0 \tag{68}$$

$$W^2 + W' = \tilde{W}^2 - \tilde{W}' + \hat{\Delta} \tag{69}$$

$$W + \tilde{W} = 2F \tag{70}$$

$$B = W\tilde{W} - \tilde{W}' = 2WF + 2F' \tag{71}$$

with solution

$$\hat{\Delta} = -\xi^2 \qquad \tilde{W} = \xi \qquad W = 2F - \xi$$

providing, as a consequence, the condition $[F, F'] = 0$, corresponding to the factorization of x-dependence of the matrix $F(x)$. It is easy to check that our potentials $V^{(1)}(x)$ do not satisfy these conditions and therefore are not reducible.

4. Conclusion

We have demonstrated that the SUSY approach allows us to relate as SUSY partners a dynamical matrix system where the symmetry is manifest to another matrix system where this symmetry is hidden in the sense that it is not otherwise easy to guess it (see, e.g. section 3.1). Therefore one is connecting systems with more complex dynamics to systems which are simpler or even solvable. For matrix QM this approach provides examples of dynamical (matrix) systems and associated symmetry operators: it is useful since a straightforward general investigation of symmetries of a dynamical systems is not an easy task. In particular the connection between degeneracy of levels and the existence of symmetry operators is not mandatory and needs further clarification which presumably will depend on detailed dynamical properties of the system under investigation.

This line of research is not academic but in contrast should become a useful approach to investigate nontrivial QM systems. We have restricted ourselves to one-dimensional matrix QM: the algebraic methods we develop therefore seem to be specifically suited for this type of dynamical systems for which spatial symmetries (such as O(3)) have already been used to reduce the problem to a one-dimensional (radial) problem (separation of variables).

In the absence of general theorems we have studied first- and second-order intertwining relations between matrix one-dimensional Hamiltonians discussing reducible and irreducible transformations among them. The examples we have discussed can be interpreted as coupled channel problems or Pauli-type Hamiltonians and our techniques may be instrumental to their diagonalization. We have provided for the first time explicit examples of irreducible transformations in the context of transparent matrix potentials in the framework of matrix SUSY QM.

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References

[1] Witten E 1981 *Nucl. Phys.* B **188** 513
Witten E 1982 *Nucl. Phys.* B **202** 253
Lahiri A, Roy P K and Bagghi B 1990 *Int. J. Mod. Phys.* A **5** 1383
Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **25** 268

[2] Andrianov A A, Borisov N V and Ioffe M V 1984 *Teor. Mat. Fiz.* **61** 183

[3] Cannata F and Ioffe M V 1993 *J. Phys. A: Math. Gen.* **26** L89
Cannata F and Ioffe M V 1992 *Phys. Lett.* B **278** 399
Fukui T 1993 *Phys. Lett.* A **178** 1
Amado R D, Cannata F and Dedonder J P 1990 *Phys. Rev.* C **41** 1289
Amado R D, Cannata F and Dedonder J P 1990 *Int. J. Mod. Phys.* A **5** 3401 and references therein

[4] Andrianov A A, Ioffe M V and Spiridonov V P 1993 *Phys. Lett.* A **174** 273
Andrianov A A, Cannata F, Dedonder J P and Ioffe M V 1995 *Int. J. Mod. Phys.* A **10** 2683
Andrianov A A, Cannata F and Ioffe M V 1996 *Mod. Phys. Lett.* A **11** 1417

[5] Andrianov A A, Ioffe M V and Nishnianidze D N 1995 *Phys. Lett.* A **201** 103

[6] Kostelecky V A and Nieto M M 1988 *Phys. Rev.* A **38** 4413

[7] Moreno M, Mendez-Moreno R M, Orozco S, Ortiz M A and de Llano M 1995 *Phys. Lett.* A **208** 113

[8] Cannata F and Ferrari L 1991 *Phys. Rev.* B **44** 8599

[9] Andrianov A A and Ioffe M V 1988 *Phys. Lett.* B **205** 507

[10] Schrödinger E 1940 *Proc. R. Irish Acad.* A **46** 9
Schrödinger E 1940 *Proc. R. Irish Acad.* A **46** 183
Schrödinger E 1940 *Proc. R. Irish Acad.* A **47** 53
Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21